Ekonomi / Matematik

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1 Rates of Change in the Natural and Social Sciences

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change. Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

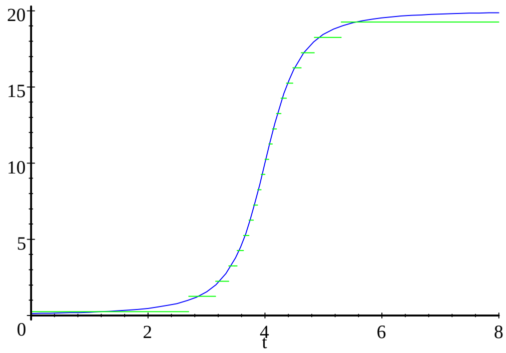
Example 1 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$ and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The instantaneous rate of growth is obtained from this average rate of growth by letting the time period $t_2 - t_1$ approach 0.

$$\lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Of course the actual growth is not continuous because populations increase and decrease by integer multiples. However, for a large animal or plant population, you can replace the graph by a smooth approximating curve as depicted in the following plot.



Definition 1 The derivative of a function f at a number x, denoted by $f\ '(x)$, $\frac{d}{dx}f\,(x)$, $\frac{df}{dx}\,(x)$, or $D_x\,(f(x))$ is the limit

$$f\ '(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists. This associates with a function $y = f(x)$, a new function $\frac{dy}{dx} = f\ '(x)$, called the derivative of f.

1.1 Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function.

1.2 The Velocity Problem

How is "instantaneous velocity" defined? If you measure the time taken to move from one point to another you can calculus the average velocity by dividing the distance traversed by the time. You can approximate the instantaneous velocity by fixing the starting point measuring the time taken to travel to successively closer points.

This is analogous to the tangent problem posed above, where you could find the slope of secant lines—lines that passed through two points on the curve—and approximate the slope of a tangent that passed through just one point.

1.3 Zooming In for a Closer Look at the Problem

To shed light on the idea of approximating a curve by a straight line, try plotting a curve and zooming in with a computer graphing system or your graphing calculator for a closer and closer look at the curve around the point in question. You will quickly see that when you get close enough to the point the curve appears as a straight line. This gives you a visual glimpse of the slope of the tangent line at that point.

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2 Functional Forms and Differentiation Rules

2.1 Quadratic Equations

If

$$ax^2 + bx + c = 0,\ a \neq 0$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2.2 Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let $x > 0$, $y > 0$, and r be any real number.

- $\log_a(xy) = \log_a x + \log_a y$

The logarithm of a product of numbers is the sum of the logarithms of the numbers.

- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.

- $\log_a (x^r) = r \log_a x$

The logarithm of a power of a number is the exponent times the logarithm of the number.

2.3 Differentiation Rules

In this table, f, g, u, v, and y represent functions of x, and c and n represent constants.

1.	$c' = 0$	$\frac{d}{dx}(c) = 0$	
2.	$(cf)' = cf'$	$\frac{d}{dx}(cu) = c\frac{du}{dx}$	
3.	$(f+g)' = f' + g'$	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$	(sum rule)
4.	$(f-g)' = f' - g'$	$\frac{d}{dx}(u-v)(x) = \frac{du}{dx} - \frac{dv}{dx}$	
5.	$(x^n)' = nx^{n-1}$	$\frac{d}{dx}(x^n) = nx^{n-1}$	(power rule)
6.	$(f(x)^n)' =$	$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$	(generalized power rule)
7.	$(fg)' = f'g + fg'$	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	(product rule)
8.	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	(quotient rule)
9.	$(f \circ g)' = (f' \circ g)\,g'$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	(chain rule)

2.4 Antiderivatives

Many problems in mathematics and its applications require the solution of the inverse of the derivative problem: given a function f, find a function F whose derivative is f. If such a function F exists, it is called an antiderivative of f.

Definition 2 A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 1 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

3 Tables: Mathematical Symbols

3.1 Numbers, Logarithms

a^n	nth power of a
$\sqrt{a}$ or $a^{\frac{1}{2}}$	square root of a
$\sqrt[n]{a}$ or $a^{\frac{1}{n}}$	nth root of a
$\log_{10}$	Common logarithm, Log base 10
ln or $\log_e$	Natural logarithm, Log base e
log	Common or natural logarithm (in context)
e	Base of natural logarithm (≈ 2.7182818)
π	Pi (≈ 3.1415927)
i or j	$\sqrt{-1}$, Imaginary number
$a + ib$ or $a + jb$	Complex number or vector
$n!$	n factorial $= 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$
$P(x, y)$	Rectangular coordinates of point P
$P(r, \theta)$	Polar coordinates of point P

3.2 Functions, Derivatives and Integrals

$f(x)$, $F(x)$, or $\varphi(x)$	Function of x
Δy	Increment of y
$\rightarrow$	Approaches as a limit
$\sum$	Summation of
dy	Differential of y
$\frac{dy}{dx}$ or $f'(x)$	Derivative of $y = f(x)$ with respect to x
$\frac{d^2y}{dx^2}$ or $f''(x)$	Second derivative of $y = f(x)$ with respect to x
$\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$	nth derivative of $y = f(x)$ with respect to x
$\frac{\partial y}{\partial t}$	Partial derivative of $y = f(s, t)$ with respect to t
$\frac{\partial^2 y}{\partial s \partial t}$	2nd partial derivative of $y = f(s, t)$ with respect to s and t
$\int f(x)dx$	Integral of $y = f(x)$ with respect to x
$\int_a^b f(x)dx$	Integral of $y = f(x)$ with respect to x between the limits a and b

3.3 Greek Alphabet

Letters		Names	Letters		Names	Letters		Names
A	α	Alpha	I	ι	Iota	P	ρ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
Δ	δ	Delta	M	μ	Mu	Υ	υ	Upsilon
E	ε	Epsilon	N	ν	Nu	Φ	ϕ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
M	η	Eta	O	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega